A visual interaction consensus model for social network group decision making with trust propagation

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Abstract

A theoretical visual interaction framework to model consensus in social network group decision making (SN-GDM) is put forward with following three main components: (1) construction of trust relationship; (2) trust based recommendation mechanism; and (3) visual adoption mechanism. To do that, dual trust propagation is investigated to connect incomplete trust relationship by trusted third partners, in a way that it can fit our intuition in these cases: trust values decrease while distrust values increase. Trust relationship is proposed to be used in determining the trust degree of experts and in aggregating individual opinions into a collective one. Three levels of consensus degree are defined and used to identify the inconsistent experts. A trust based recommendation mechanism is developed to generate advices according to individual trust relationship, making recommendations more likeable to be implemented by the inconsistent experts to achieve higher levels of consensus. Therefore, it has an advantage with respect to existing interaction models because it does not force the inconsistent experts to accept advices irrespective of their trust on them. Finally, a visual adoption mechanism, which provides visual information representations on experts' individual consensus positions before and after adopting the recommendation advices, is presented and analysed theoretically. Experts can select their appropriate feedback parameters to achieve a balance between group consensus and individual independence. Consequently, the proposed visual interaction model adds real and needed flexibility in guiding the consensus reaching process in SN-GDM.

Keywords: Social network group decision making, Visual interaction, Consensus, Trust recommendation, Adoption mechanism, Trust propagation

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1. Introduction

In group decision making (GDM), a group of experts express preference values (opinions) on alternatives and interact to derive a common solution [14, 22]. Group experts usually come from

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multiple organisations with different backgrounds and knowledge on the decision making problem faced, which may lead to inconsistency problems among the group [5, 29, 30, 35, 36, 39, 43, 48]. Thus, how to reach consensus is relevant and a hot topic that has attracted the interest of researchers in the field of GDM [2, 12, 17, 42]. The group interaction consensus model has been proved to be an effective method to increase consensus within a group of experts because it supports inconsistent experts, i.e. experts with a consensus level below a target threshold value, with advices on how to modify their preference values [23, 47]. However, these known group interaction consensus models do not take into account the trust relationship among the experts in the group [14, 20, 44], which makes them not suitable for GDM problems in a social network framework where the relationships between the members of a group is present and should be considered in obtaining a solution of consensus [13, 28, 33].

A new trend of GDM today is that individuals rely on the opinions and social appraisal support from their close friends or people with similar interests, and then the social relationship should be taken into account in the interaction process of GDM [1, 19, 21, 24, 25, 27]. An example is the community review website Ciao (www.ciao.co.uk), where users can write reviews about consumer products and assign a rating to the products and the reviews written by others. In essence, Ciao is a social network, i.e., “an association of people drawn together by family, work or hobby”. By reviewing and talking, consumers can build a trust relationship between them, which in turn can be used to reach higher consensus level in the group interaction. Then, the trust relationship should be considered an important factor influencing the consensus process in GDM as pointed by Herrera-Viedma et al. [16]. This social network group decision making (SN-GDM) would be a completely new type of decision making because it takes into account the trust relationship in the process of reaching consensus while the traditional GDM models neglect it. Consequently, key issues in SN-GDM problems would be: (1) the study and modelling of trust relationship between experts, and (2) how to reach group consensus under trust relationship.

In a realistic trust network, there exist indirect trust relationships between experts that are unknown to each other as Fig. 1 illustrates. For such cases, there is a need to devise a mechanism to find out whether or not an unknown expert can be trusted [38]. To do that, a trust chain via trusted third partners (TTPs) can be built to propagate trust to an unknown expert. This will obviously require the development of appropriate trust propagation operators. Victor et al. [31] have proposed t-norms to propagate trust and t-conorms to propagate distrust, while Wu et al. [41] developed a uninorm operator to propagate both trust and distrust at the same time. However, these operators share the limitation of decreasing both the trust and distrust values simultaneously, which conflicts with our intuition because the trust propagating process using trusted third parters (TTP) may produce information attenuation that makes trust values to decrease but distrust values to increase. To avoid this issue, this article will investigate a novel dual trust propagation operator based on the t-norm Einstein product and the t-conorm Einstein sum. In addition, its desirable properties including 'T-subnorm' and 'T-upconorm' can be guaranteed.

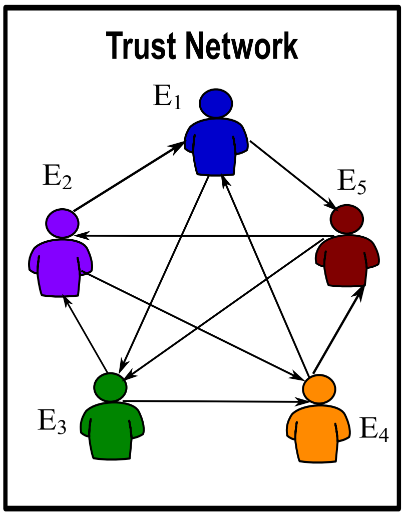


Figure 1: Trust Network: solid arrows mean presence of direct trust

As aforementioned, the group interaction models with a recommendation mechanism were regarded as an effective method to reach satisfied consensus level in traditional GDM [14, 44]. However, they usually generate recommendation advices for the inconsistent experts based on the arithmetic average of the individual opinions of the experts in the group with no consideration at all on their trust relationship. As such, in the known group interaction models inconsistent experts have no idea whether the change of opinion recommendation advice given can be trusted/distrusted. This is not ideal and it is indeed in conflict with real decision making cases because it is up to the experts to adopt or not the recommendation advices provided to him/her [9, 34]. Additionally, known group interaction models share another limitation associated to the fixation of a feedback parameter, which decreases the adaptability of the recommendation advices to inconsistent experts' willingness of change their individuals original opinions. Therefore, they might wish to keep unaltered their independence [3, 7, 37, 40] or, if possible, to reduce changes cost [6, 8, 10, 11]. In other words, experts are willing to keep balance between group consensus and individual independence.

To resolve the aforementioned issues, this article firstly investigates a trust based recommendation mechanism to generate recommendation advices by trust relationship obtained via TTPs. This will surely contribute to inconsistent experts willingness to adopt or implement the advices received. Secondly, a visual adoption mechanism in which the consensus status is simulated by choosing different feedback parameter is developed. The visual representations of information can help experts to analyse their consensus position before and after adopting the advices received. Consequently, the proposed visual group interaction model will allow experts to decide by themselves on selecting the right feedback parameter to balance consensus and independence. Obviously, this new approach offers more flexibility than the known group interaction models, and so it is suitable for SN-GDM contexts.

The rest of this paper is set out as follows: Section 2 introduces the definition of dual trust propagation operator and studies its desirable properties. The complete trust relationship of group experts is also constructed in this section. In Section 3, the concept of trust degree (TD) is defined and used to aggregate the individual trust decision making matrices into a collective one. Section 4 presents the novel visual interaction model for consensus in SN-GDM. In detail, it includes the trust based recommendation mechanism and the visual adoption mechanism. The framework of the visual interaction consensus model proposed including all steps of the decision process is provided in Section 5. Section 6 uses a numerical example to illustrate the proposed methods. Finally, conclusions are drawn in Section 7.

2. Dual Trust Propagation and Aggregation in Social Network

Social Network Analysis (SNA) [13, 28, 33] studies the relationships between social entities like members of a group, corporations or nations. Therefore, it enables us to examine the structural and locational properties including centrality, prestige, structural balance, trust relationship etc. There are three notational schemes in SNA analysis:

- Graph, as in Fig. 1, where the network is viewed as a set of nodes joined by lines (edges).
- Algebraic, which allows to distinguish several distinct relations and to represent combinations of relations. The corresponding algebraic representation of Fig. 1 would be:

  $$\begin{array}{ccccc} E_1RE_3 & E_3RE_4 & E_4RE_5 & E_5RE_2 & E_2RE_4 \\ E_4RE_1 & E_1RE_5 & E_5RE_3 & E_3RE_2 & E_2RE_1 \end{array}$$

  where E_iRE_j represents the existence of a directed graph edge from node E_i to node E_j.
- Sociometric, which presents relational data in a two-ways matrix called sociomatrix. The corresponding sociomatrix representation of Fig. 1 would be:

  $$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The above sociomatrix represents a binary complete relation. This type of sociomatrix has two main limitations:

1. It is not suitable to model uncertainty associated to the relationship representation in social network as it is the case with trust, which is indeed often interpreted as a gradual phenomenon: humans do not merely reason in terms of ‘trusting’ and ‘not trusting’, but rather trusting someone ‘very much’ or ‘more or less’ [4].

2. Binary trust relationships cannot be propagated and as such indirect trust relationships cannot be built.

To overcome these problems, this article adopts one type of social networks, namely trust network in which the users explicitly express their opinion as trust and distrust statements [32].

***Definition* 1 (Trust Function (TF)).** A tuple of the type $\lambda = (t, d)$ where $t, d \in [0, 1]$, in which the first component t is a *trust degree*, and the second component d is a *distrust degree* will be referred to as a trust function value. The set of trust function values (TFs), or trust function, will be denoted by $\Lambda = \{\lambda = (t, d) |\ t, d \in [0, 1]\} \equiv [0, 1]^2$.

The concepts of trust score and knowledge degree were proposed to define an order relation in the set of TFS [41]:

***Definition* 2 (Trust Score (TS)).** The trust score is a mapping on the set of TFs, Λ, that associates a value in $[0, 1]$ to each trust function value λ as follows:

$$TS \colon \Lambda \longrightarrow [0, 1]$$

$$TS(\lambda) = \frac{t - d + 1}{2} \tag{1}$$

***Definition* 3 (Knowledge Degree (KD)).** The knowledge degree is a mapping on the set of TFs, Λ, that associates a value in $[0, 1]$ to each trust function value λ as follows:

$$KD \colon \Lambda \longrightarrow [0, 1]$$

$$KD(\lambda) = (1 - t - d)^2 \tag{2}$$

***Definition* 4 (Order Relation of TFs).** Given two TFs, λ_1 and λ_2, λ_1 precedes λ_2

$$\lambda_1 \prec \lambda_2$$

if and only if one of the following conditions is true:

1. $TS(\lambda_1) < TS(\lambda_2)$
2. $TS(\lambda_1) = TS(\lambda_2)\ \wedge\ KD(\lambda_1) > KD(\lambda_2)$

The above order relation of TFs can be used to distinguish the most trusted expert from the group, which in turn can be useful in performing an heterogeneous aggregation of individual opinions by establishing the ordering of the decision matrix values to be aggregated.

2.1. Trust propagation

Trust propagation operators have been proposed and built using the general concepts of the triangular norms and triangular conorms. A function $T\colon [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm for short) if and only if it is commutative, associative, monotonic and satisfies the following boundary conditions $T(x,1) = x,\ \forall x$. A function $S\colon [0,1]^2 \to [0,1]$ is called a triangular conorm (t-conorm for short) if and only if it is commutative, associative, monotonic and satisfies the following boundary conditions $S(x,0) = x,\ \forall x$. In the following, we will be making use of the Einstein product, $\otimes_\varepsilon$, and Einstein sum, $\oplus_\varepsilon$ as the t-norm and a t-conorm [18], respectively, to illustrate the general approach to trust propagation proposed in this paper:

$$E_\otimes = a \otimes_\varepsilon b = \frac{a \cdot b}{1 + (1-a)\cdot(1-b)}, \forall\,(a,b) \in [0,1]^2 \tag{3}$$

$$E_\oplus = a \oplus_\varepsilon b = \frac{a+b}{1 + a\cdot b} \tag{4}$$

It is worth mentioning that the greatest of all t-norms is the minimum operator, while the lowest of all t-conorm is the maximum operator. Consequently, we have that

$$E_\otimes(x_1, x_2) \le \min\{x_1, x_2\} \text{ (t-subnorm)} \tag{5}$$

$$\max\{x_1, x_2\} \le E_\oplus(x_1, x_2) \text{ (t-upconorm)} \tag{6}$$

The above definitions just present the $E_\oplus$ and $E_\otimes$ as aggregation operators of two arguments. However, the associativity property allows their use with n arguments:

$$E_\oplus(x_1, \ldots, x_n) = \frac{\prod_{i=1}^n (1+x_i) - \prod_{i=1}^n (1-x_i)}{\prod_{i=1}^n (1+x_i) + \prod_{i=1}^n (1-x_i)} \tag{7}$$

and

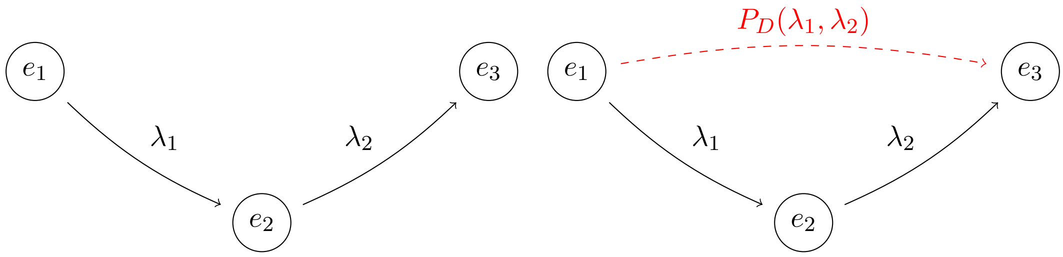
$$E_\otimes(x_1, \ldots, x_n) = \frac{2\prod_{i=1}^n x_i}{\prod_{i=1}^n (2-x_i) + \prod_{i=1}^n x_i} \tag{8}$$

The monotonocity property implies that the t-subnorm and t-upconorm properties also hold for the case on of n arguments:

$$E_\otimes(x_1, x_2, \ldots x_n) \le \min(x_1, x_2, \ldots x_n)$$

$$\max(x_1, x_2, \ldots x_n) \le E_\oplus(x_1, x_2, \ldots x_n)$$

Existing trust propagation operators do not take into account the distrust value of TTP in the trust chain [31, 41]; additionally they also have one main limitation in that both trust and distrust values decrease simultaneously. Considering the information attenuation in a realistic trust propagating process via a TTP, a reasonable policy should be that the trust value decreases and the distrust value increases as a result of a propagation process. To achieve this aim, we propose a dual trust propagation operator by combining the Einstein sum operator $E_\oplus$ and the Einstein product operator $E_\otimes$ (see Fig. 2).



(a) No direct TF between e_1 and e_3 (b) Trust propagation between e_1 and e_3 via e_2

Figure 2: Dual trust propagation via indirect chain of TFs

***Definition* 5.** Let Λ be the set of TFs. The dual trust propagation operator P_D is mapping, $P_D : \Lambda \times \Lambda \to \Lambda$, that associates to two TFs $\lambda_1 = (t_1, d_1), \lambda_2 = (t_2, d_2)$, the following TF output

$$P_D(\lambda_1, \lambda_2) = (E_{\otimes}(t_1, t_2), E_{\oplus}(d_1, d_2)) = \left(\frac{t_1 t_2}{1 + (1 - t_1)(1 - t_2)}, \frac{d_1 + d_2}{1 + d_1 \cdot d_2} \right) \tag{9}$$

The following observations are noticed:

1. Because $E_{\otimes}(t_1, t_2) \leq \min\{t_1, t_2\}$ and $\max\{d_1, d_2\} \leq E_{\oplus}(d_1, d_2)$, the dual trust propagation operator P_D verifies that the trust value and distrust value decreases and increases, respectively, as a result of the propagation process.

2. P_D inherits the commutativity and associativity properties of $E_{\oplus}$ and $E_{\otimes}$.

   Commutativity:

   $$P_D(\lambda_2, \lambda_1) = (E_{\otimes}(t_2, t_1), E_{\oplus}(d_2, d_1)) = (E_{\otimes}(t_1, t_2), E_{\oplus}(d_1, d_2)) = P_D(\lambda_1, \lambda_2)$$

   Associativity:

   $$\begin{aligned} P_D[P_D(\lambda_1, \lambda_2), \lambda_3] &= P_D[(E_{\otimes}(t_1, t_2), E_{\oplus}(d_1, d_2)), (t_3, d_3)] = (E_{\otimes}(E_{\otimes}(t_1, t_2), t_3), E_{\oplus}(E_{\oplus}(d_1, d_2), d_3)) \\ &= (E_{\otimes}(t_1, E_{\otimes}(t_2, t_3)), E_{\oplus}(d_1, E_{\oplus}(d_2, d_3))) = P_D[(t_1, d_1), (E_{\otimes}(t_2, t_3), E_{\oplus}(d_2, d_3))] \\ &= P_D[\lambda_1, P_D(\lambda_2, \lambda_3)] \end{aligned}$$

3. **Monotonicity Criteria** for P_D. Because $E_{\oplus}$ and $E_{\otimes}$ are monotonic, we have that P_D does not decrease (increase) the trust [distrust] value when the chain trust [distrust] values increase (decrease).

4. **Boundary Conditions** for P_D. we have:

   Full trust propagation: When $\lambda_1 = (1, 0)$ we have:

   $$P_D((1, 0), \lambda_2) = (E_{\otimes}(1, t_2), E_{\oplus}(0, d_2)) = (t_2, d_2) = \lambda_2.$$

   When $\lambda_2 = (1, 0)$, commutativity property implies that

   $$P_D(\lambda_1, (1, 0)) = P_D((1, 0), \lambda_1) = \lambda_1.$$

Thus, in a chain of three experts, if one expert fully trusts another expert, the TF of first expert to the third expert in the chain will be equal to the TF between the two other experts.

Full distrust propagation: When $\lambda_1 = (0, 1)$ we have:

$$P_D((0,1), \lambda_2) = (E_{\otimes}(0, t_2), E_{\oplus}(1, d_2)) = (0, 1).$$

When $\lambda_2 = (0, 1)$, commutativity property implies that

$$P_D(\lambda_1, (0,1)) = P_D((0,1), \lambda_1) = (0, 1).$$

Thus, in a chain of three experts, if one expert fully distrusts another expert, then the first expert will fully distrust the third expert in the chain no matter what the TF between the other two expert is.

Boundary consistency. In a chain of three experts in which the first expert fully trusts the second expert and this second expert fully distrusts the third expert, both above boundary conditions result in the same fully distrusts TF output, and therefore they are consistent with each other.

Example 1. Assume the following two TFs $\lambda_1 = (0.8, 0.4)$ and $\lambda_2 = (0.7, 0.5)$. Applying the dual trust propagation operator in expression (9), we obtain

$$P_D(\lambda_1, \lambda_2) = (0.53, 0.75)$$

Using the propagating operator proposed by Victor et al. [29], we obtain

$$P_V(\lambda_1, \lambda_2) = (t_1 t_2, t_1 d_2) = (0.56, 0.4)$$

Victor et al.'s trust propagating process makes the distrust value to decrease, which is not what is expected in real situations. Another limitation is that it does not take into account the distrust value in λ_1, and therefore monotonicity is not satisfied under Victor et al.'s trust propagating process. Indeed, if one of the distrust value is modified and $\lambda_1 = (0.8, 0.3)$ and $\lambda_2 = (0.7, 0.5)$, we obtain that:

$$P_D(\lambda_1, \lambda_2) = (0.53, 0.70)$$

and

$$P_V(\lambda_1, \lambda_2) = (t_1 t_2, t_1 d_2) = (0.56, 0.4)$$

It is clear that the proposed dual trust propagating operator P_D can reflect the change of the distrust value in λ_1 while P_V cannot.

In real decision making, a trust propagating chain may involve more than three experts. For example, in Fig. 1 there are two indirect paths from E_1 to E_2 that involve one other expert [E_3 and

E_5, respectively]; but also one path that involves three other experts [(E_3, E_4, E_5)]. In this last case, associativity property of P_D can be applied and the following is obtained:

$$P_D\left((t_1, d_1), (t_2, d_2), \ldots, (t_n, d_n)\right) = \left(E_{\otimes}(t_1, t_2, \ldots, t_n), E_{\oplus}(d_1, d_2, \ldots, d_n)\right) \quad (10)$$

Expressions (7) and (8) yield:

$$P_D\left((t_1, d_1), (t_2, d_2), \ldots, (t_n, d_n)\right) = \left(\frac{2\prod_{i=1}^{n} t_i}{\prod_{i=1}^{n}(2 - t_i) + \prod_{i=1}^{n} t_i}, \frac{\prod_{i=1}^{n}(1 + d_i) - \prod_{i=1}^{n}(1 - d_i)}{\prod_{i=1}^{n}(1 + d_i) + \prod_{i=1}^{n}(1 - d_i)}\right) \quad (11)$$

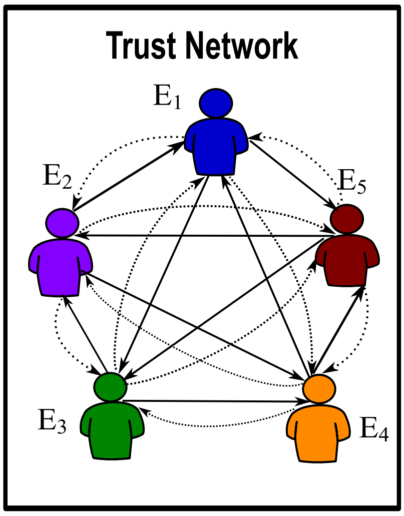


Figure 3: Fully Connected Trust Network after propagation: dash arrows means absence of direct trust computed via indirect TTPs

Example 2. A company invites a group of experts $(e_1, e_2, e_3, e_4, e_5)$ from five different departments to select the most appropriate cloud service supplier. They have the following a priori trust relationship as depicted in Figure 1, and corresponding incomplete trust sociomatrix T_L:

$$T_L = \begin{pmatrix} - & - & (0.6, 0.1) & - & (0.8, 0.1) \\ (0.8, 0.5) & - & - & (0.7, 0.5) & - \\ - & (0.7, 0.2) & - & (0.5, 0.2) & - \\ (0.8, 0.3) & - & - & - & (0.6, 0.3) \\ - & (0.8, 0.5) & (0.3, 0.1) & - & - \end{pmatrix}$$

To fully connect the network with trust links between all the nodes as Figure 3 illustrates, indirect TTPs (Fig. 2(a)) to propagate TFs via dual trust propagation operation P_D is used (Fig. 2(b)). For example, to propagate TFs from expert E_1 to expert E_2, there are five possible indirect TTPs paths: $L_1 : E_1 \rightarrow E_3 \rightarrow E_2$; $L_2 : E_1 \rightarrow E_5 \rightarrow E_2$; and $L_3 : E_1 \rightarrow E_3 \rightarrow E_4 \rightarrow E_5 \rightarrow E_2$. To reduce attenuation of information, only shortest indirect paths are used, and when more than one the average of the propagated TFs is computed. In this case, L_1 and L_2 are used to propagate TFs from E_1 to E_2. The computation involved is:

$$P_D^{L_1}\left((0.6, 0.1), (0.7, 0.2)\right) = (0.375, 0.294);$$

$$P_D^{L_2}\left((0.8, 0.1), (0.8, 0.5)\right) = (0.615, 0.574)$$
$$P_D^{E_1 \to E_2} = \frac{P_r^{L_1} + P_r^{L_2}}{2} = (0.5, 0.43)$$

The completed trust sociomatrix would be:

$$T_L = \begin{pmatrix} - & (0.50, 0.43) & (0.60, 0.10) & (0.25, 0.29) & (0.80, 0.10) \\ (0.80, 0.50) & - & (0.44, 0.57) & (0.70, 0.50) & (0.50, 0.63) \\ (0.53, 0.64) & (0.70, 0.20) & - & (0.50, 0.20) & (0.25, 0.47) \\ (0.80, 0.30) & (0.44, 0.70) & (0.29, 0.39) & - & (0.60, 0.30) \\ (0.62, 0.80) & (0.80, 0.50) & (0.30, 0.10) & (0.32, 0.55) & - \end{pmatrix}$$

3. Trust Based Aggregation of Decision Matrices

3.1. Trust Score of Experts

When the complete trust relationship is constructed, each expert can be associated with a TS as follows [38]:

***Definition* 6.** Let $G = (E, L, \omega)$ be a directed graph, $E = \{e_1, \ldots, e_k\}$ be the set of nodes and $L = \{l_1, \ldots, l_q\}$ be the set of directed lines, or arcs, between pairs of nodes and $S_L = (S_{lh})_{k \times k}$ be the trust sociomatrix associated with the graph $G = (E, L, \omega)$, where S_{lh} is the trust function from expert E_l to expert E_h. Then

- The TF of expert E_h is:
$$(t_h, d_h) = \frac{1}{k-1} \sum_{l=1}^{k} S_{lh} \tag{12}$$
- The TS of expert E_h is :
$$TS_h = \frac{t_h - d_h + 1}{2} \tag{13}$$

3.2. Importance Weights by Trust Score

TSs values can be used to assign importance weights to experts. Using Yager's OWA based procedure, experts importance weight are computed as follows [45, 46]:

$$w_T^{\sigma(h)} = Q\left(\frac{T(\sigma(h))}{T(\sigma(k))}\right) - Q\left(\frac{T(\sigma(h-1))}{T(\sigma(k))}\right) \tag{14}$$

with $T(\sigma(h)) = \sum_{l=1}^{h} TS_{\sigma(l)}$, σ is a permutation such that $TS_{\sigma(l)}$ is the $l-$th largest value of set $\{TS_1, \ldots, TS_k\}$ and Q is a Basic Unit-interval Monotone (BUM) membership function of the fuzzy linguistic quantifier to implement in the aggregation process: $Q : [0, 1] \to [0, 1]$ such that $Q(0) = 0$, $Q(1) = 1$ and if $x > y$ then $Q(x) \geq Q(y)$.

3.3. Trust Based Collective Decision Matrix

The individual decision matrices are aggregated into a collected one using the above TS based weights as follows:

Definition 7. Let $\{R^{(h)} = (r_{ij}^{(h)})_{m\times n}; h = 1, 2, \ldots, k\}$ be a collection of decision matrices given by a set of experts $E = \{e_1, \ldots, e_k\}$ with associated TSc $\{TS_1, \ldots, TS_k\}$. The collective decision matrix guided by trust is $\overline{R} = (\overline{r}_{ij})_{m\times n}$ with element

$$\overline{r}_{ij} = \sum_{h=1}^{k} w_T^h \cdot r_{ij}^{(h)},\ i = 1, \ldots, m; j = 1, \ldots, n \tag{15}$$

where w^h given by expression (14).

4. Visual Interaction Method for Group Consensus

Once the trust based collective decision matrix is obtained, consensus indexes (CI) at three levels are computed for each expert: (1) elements level; (2) alternatives level; and (3) decision matrix level. When the decision matrix level CI (group CI) reaches a set threshold value for all experts, the resolution process of the SN-GDM is carried out; otherwise the inconsistent experts are identified and invited to revisit some of their opinions that contribute less to consensus in order to increase the group CI. This paper proposes to do this by using a visual interaction method to provide expert with a visual representations of his/her consensus status. Additionally, a trust based recommendation mechanism is also developed to advice inconsistent experts, i.e experts with a lower consensus contribution than the set threshold value, on how to change their identified evaluation contributing less to consensus. This is complemented with a visual adoption mechanism to allow experts to 'see' their future consensus position if the recommendation advices were to be implemented. Using this visual extra information, the inconsistent experts can achieve a balance between group consensus and individual independence.

The visual interaction for SN-GDM with trust relationship is depicted in Figure 4. Specifically, it consists of the following five steps:

(1) Developing the dual trust propagation operator;

(2) Computing TS of experts and trust collective decision matrix;

(3) Determining CI at three levels;

(4) Visual consensus identification, trust based recommendation and visual adoption mechanism; and

(5) Selection Process.

The first and second steps have already been covered in Section 2 and Section 3, respectively. The remaining steps will be presented in more detail in the following subsections.

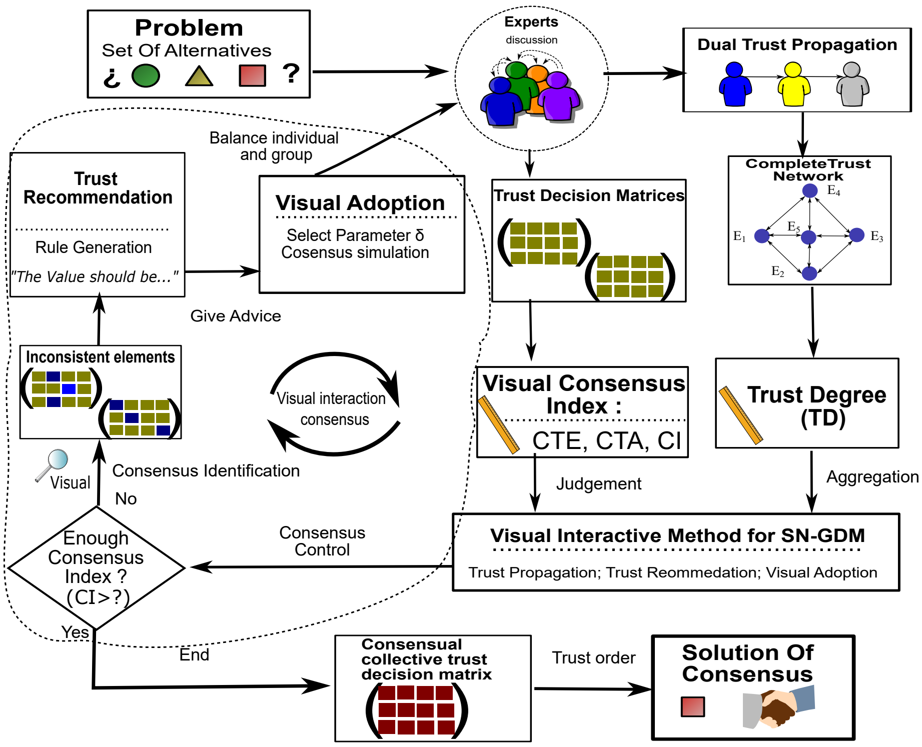


Figure 4: Visual interaction consensus model for SN-GDM with trust propagation

4.1. Visual Identification of Three Consensus Levels

The visual consensus model consists of three consecutive stages: (1) visual identification of the preference values with a consensus level lower than the set threshold value; (2) recommendation simulation for generating advice on the direction-value of the required change to increase current consensus; (3) visual recommendation process simulation to show what would happen if experts are to accept the recommended preference values.

First, the consensus index for each expert at the three different levels of a relation are computed as follow:

Level 1. *Consensus index at element levels (pair of alternative-criterion).* The consensus index of an expert e_h with respect to the group on the alternatives x_i under criterion c_j is

$$CTE_{ij}^h = 1 - d(r_{ij}^h, \overline{r}_{ij}) = 1 - \frac{\left|t_{ij}^h - \bar{t}_{ij}\right| + \left|d_{ij}^h - \bar{d}_{ij}\right|}{2} \tag{16}$$

Level 2. *Consensus index at alternatives level.* The consensus index of an expert e_h with respect to the group on the alternative x_i is

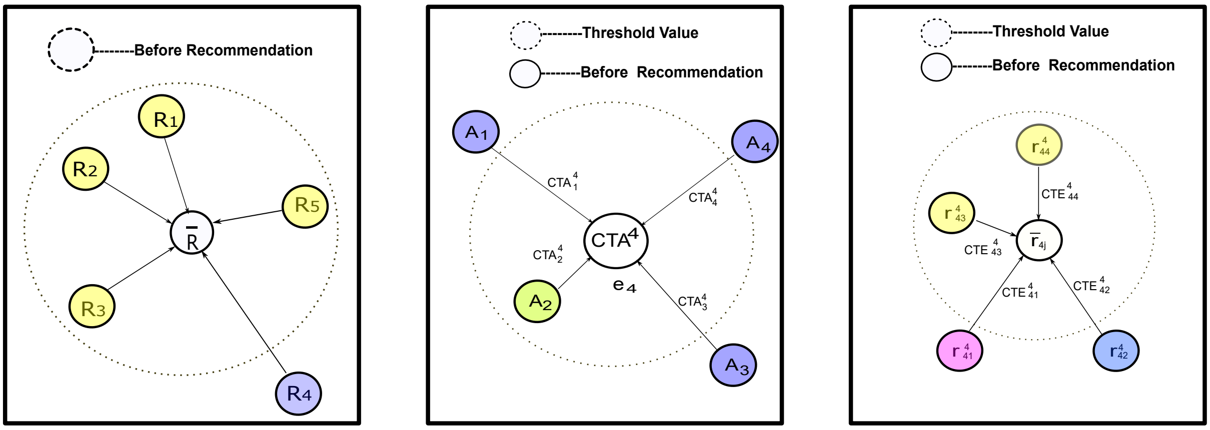
$$CTA_i^h = \frac{1}{n}\sum_{j=1}^{n} CTE_{ij}^h \tag{17}$$

Level 3. *Consensus index at decision matrix level.* The consensus index of an expert e_h to the group on decision matrix is

$$CI^h = \frac{1}{m}\sum_{i=1}^{m} CTA_i^h \tag{18}$$

The greater the value of $CI^h (0 \leq CI^h \leq 1)$, the greater the consensus index between individual expert e_h and the group. When $CI^h = 1$, this means that expert e_h has same decision matrix as the group collective decision matrix (unanimous agreement). However, this case is rare in practice and possibly impractical nor desirable in a decision making context. Consequently, a consensus threshold γ such as $90\%, 80\%$, $two - thirds$ are commonly set as a minimum level to achieve [26]. When at least a CI^h is less than the predefined threshold γ, a visual interaction procedure is activated to help inconsistent experts see where they are and what can be changed to increase their consensus with respect to the group. Otherwise, an appropriate selection process is applied to derive the solution of consensus.

The visual identification uses visual graphs to help experts 'see' their consensus position within the group as illustrated in Figure 5. Using the threshold value $\gamma = 0.9$, Figure 5(a) presents a visual representation of all experts consensus level with respect to the threshold value. It is clear that e_4 is identified as the only expert contributing less to group consensus. Additionally, individual visual representations of consensus indexes at alternatives and elements levels, respectively, are also provided to the identified experts at the decision matrix level to help them see in which alternatives and in which preference values he/she is contributing less to consensus. Mathematically, these steps are modelled as follows:



(a) The visual consensus index at the decision matrix level:

(b) The visual consensus index at the alternatives level for expert e_4

(c) The visual consensus index at the element A_4 level for expert e_4

Figure 5: Three visual levels of consensus before adopting the recommendation advice

Step 1. Experts with a consensus index at decision matrix lower than the threshold value γ are identified:

$$EXPCH = \{h \mid CI^h < \gamma\}$$

Step 2. For the identified experts in Step 1, their alternatives with a consensus index CTA_i^h lower than the threshold γ are identified:

$$ALT = \{(h,i) \mid h \in EXPCH \ \wedge \ CTA_i^h < \gamma\}$$

Step 3. Finally, the preference values to be changed are those with a consensus index CTE_i^h lower than the threshold γ :

$$APS = \{(h,i,j) \mid (h,i) \in ALT \ \wedge \ CTE_{ij}^h < \gamma\}.$$

4.2. Trust based Recommendation for Generating Advice

The recommendation produces personalised advice for experts on how to modify their assessment values to increase their consensus level. Given $(h,i,j) \in APS$, expert e_h receives the following personalised advice rules:

1. *"You should change your evaluation for alternative x_i under criteria c_j, r_{ij}^h, to a value closer to rr_{ij}^h."*

$$rr_{ij}^h = (1-\delta) \cdot r_{ij}^h + \delta \cdot \overline{r}_{ij} \tag{19}$$

   where $\delta \in [0,1]$ is a parameter to control the degree of advice and $\overline{r}_{ij}$ is the trust based collective evaluation as per Definition 7.

4.3. Visual Adoption Mechanism for Consensus

Notice that when the feedback parameter δ takes value 1, the original assessment is completely replaced by the trust based collective evaluation using the above recommendation rule, while when the feedback parameter δ takes value 0 the original assessment is kept unchanged. It would be interesting to know as well how the consensus index at the elements level CTE_{ij}^h is affected by the feedback parameter δ when changes are implemented, so that experts can analyse which δ value is best to achieve an acceptable compromise between group consensus and individual independence.

Notice that the new recommended values are closer than the original values to the collective evaluation, which when fixed will imply that the consensus index at elements level would increase with the value of the parameter δ. An extreme case of this scenario would derive with the production of recommendations to all experts on decision elements for all pairs of alternative-criterion. The implementation of the recommended values will guarantee that all expert consensus index at the elements values will increase values increase. This is proved in the following:

***Lemma* 1.** *Let $\{R^h = (r^h_{ij})_{m\times n}; h = 1, 2, \ldots, k\}$ be a collection of decision matrices given by a set of experts $E = \{e_1, \ldots, e_k\}$ with associated TSc $\{TS_1, \ldots, TS_k\}$. Let $\overline{R} = (\overline{r}_{ij})_{m\times n}$ be the collective decision matrix guided by trust. Let $\{rR^h = (rr^h_{ij})_{m\times n}; h = 1, 2, \ldots, k\}$ be the set of new decision matrices with*

$$rr^h_{ij} = (1-\delta)\cdot r^h_{ij} + \delta\cdot \overline{r}_{ij};\ \ \delta \in [0,1]$$

and $\overline{rR}$ be their corresponding collective decision matrix guided by trust. Then, it is:

$$\overline{rR} = \overline{R}$$

Proof. Expression (14) is used to derive weighting vector $\{w^h_T; h = 1, 2, \ldots, k\}$ to compute the collective decision matrices as per Definition 7:

$$\begin{aligned}\overline{rr}_{ij} &= \sum_{h=1}^{k} w^h_T \cdot rr^h_{ij} = \sum_{h=1}^{k} w^h_T \cdot \left[(1-\delta)\cdot r^h_{ij} + \delta\cdot\overline{r}_{ij}\right]\\ &= (1-\delta)\cdot\sum_{h=1}^{k} w^h_T\cdot r^h_{ij} + \delta\cdot\overline{r}_{ij}\cdot\sum_{h=1}^{k} w^h_T = (1-\delta)\cdot\overline{r}_{ij} + \delta\cdot\overline{r}_{ij}\\ &= \overline{r}_{ij}\end{aligned}$$

□

***Proposition* 1.** *Under the condition of Lemma 1 we have: $d(rr^h_{ij}, \overline{rr}_{ij}) \le d(r^h_{ij}, \overline{r}_{ij})$.*

Proof. Because $d(r^h_{ij}, \overline{r}_{ij}) = \dfrac{\left|t^h_{ij} - \overline{t}_{ij}\right| + \left|d^h_{ij} - \overline{d}_{ij}\right|}{2}$, to prove that $d(rr^h_{ij}, \overline{rr}_{ij}) \le d(r^h_{ij}, \overline{r}_{ij})$ we only need to prove that $d(rt^h_{ij}, \overline{rt}_{ij}) \le d(t^h_{ij}, \overline{t}_{ij})$ and $d(rd^h_{ij}, \overline{rd}_{ij}) \le d(d^h_{ij}, \overline{d}_{ij})$. Applying Lemma 1, we have:

$$d(rt^h_{ij}, \overline{rt}_{ij}) = \left|rt^h_{ij} - \overline{rt}_{ij}\right| = \left|(1-\delta)\cdot t^h_{ij} + \delta\cdot\overline{t}_{ij} - \overline{t}_{ij}\right| = (1-\delta)\cdot\left|t^h_{ij} - \overline{t}_{ij}\right| \le \left|t^h_{ij} - \overline{t}_{ij}\right| = d(t^h_{ij}, \overline{t}_{ij})$$

Similarly, we can prove that $d(rd^h_{ij}, \overline{rd}_{ij}) \le d(d^h_{ij}, \overline{d}_{ij})$. □

Because $d(rr^h_{ij}, \overline{rr}_{ij}) = (1-\delta)\cdot d(r^h_{ij}, \overline{r}_{ij})$, we have that the feedback mechanism would make the consensus process to converge to a unanimous consensus when $\delta > 0$. As mentioned before, the above results are only valid when all experts change all their assessment values to the recommendation ones. Therefore, the production of recommendations to all experts and their implementation will guarantee that experts consensus index at elements level will increase, and consequently the group consensus index will also increase (monotonically with respect to the parameter δ). However, in real applications all experts will not receive recommendations to change values at alternative level, otherwise the computational complexity of the decision making process would increase. Therefore, in the consensus process proposed here only those experts with a consensus level lower than the threshold value will receive recommendations, and the previous results are not applicable. Indeed, the collective decision matrix when changes are implemented by just some experts on some of their values is not guaranteed to be the same than before the implementation of changes. It is expected though that those experts receiving recommendations and implementing recommended values will become closer to the new collective decision matrix than before the implementation of changes, and in turn it would lead to an increase of their consensus index at elements level. Obviously, the same conclusion cannot be drawn for the experts that do not change their original assessment values. The following result indicates that if advices are implemented by just one expert, then this expert's consensus index at elements level will increase.

***Proposition* 2.** *If only expert e_s adopts the recommended advices then the new consensus index at element level for such expert will be greater than or equal to his/her previous consensus index at element level. Furthermore, the new consensus index at element level is monotonic increasing with respect to parameter δ.*

Proof. After the expert e_s adopts recommended value for alternative x_i under criteria c_j , then assessment values for alternative x_i under criteria c_j are divided into two groups: the new preference value $\{rr_{ij}^s | rr_{ij}^s = (1-\delta)r_{ij}^h + \delta \times \overline{r}_{ij}\}$ and the set of unchanged assessment values $\{(rr_{ij}^h) | rr_{ij}^h = r_{ij}^h, h = 1, \cdots, k, h \neq s\}$. Then, we have

$$\overline{rr}_{ij} = w_T^s \cdot rr_{ij}^s + \sum_{h=1, h\neq s}^{k} w_T^h \cdot rr_{ij}^h = w_T^s \cdot rr_{ij}^s - w_T^s \cdot r_{ij}^s + \overline{r}_{ij}$$

and the new consensus index for alternative x_i under criteria c_j, which we will denote by $\overline{CTE}_{ij}$ to differentiate it from the previous one CTE_{ij}, would be

$$\begin{aligned}
\overline{CTE}_{ij}^s &= 1 - \left| rr_{ij}^s - \overline{rr}_{ij} \right| \\
&= 1 - \left| (1-\delta) \cdot r_{ij}^s + \delta \cdot \overline{r}_{ij} - \left(w_T^s \cdot rr_{ij}^s - w_T^s \cdot r_{ij}^s + \overline{r}_{ij} \right) \right| \\
&= 1 - \left| (1-\delta) \cdot r_{ij}^s + \delta \cdot \overline{r}_{ij} - \left(w_T^s \cdot \left((1-\delta) \cdot r_{ij}^s + \delta \overline{r}_{ij} \right) - w_T^s \cdot r_{ij}^s + \overline{r}_{ij} \right) \right| \\
&= 1 - [1 - \delta \cdot (1 - w_T^s)] \cdot \left| r_{ij}^s - \overline{r}_{ij} \right|
\end{aligned}$$

Because $w_T^s, \delta \in [0,1]$, it is $1 - \delta \leq 1 - \delta \cdot (1 - w_T^s) \leq 1$, and consequently we will have that the new consensus index for alternative x_i under criteria c_j for expert e_s will be greater than or equal to the previous consensus index for alternative x_i under criteria before implementing the corresponding advice valued. The increment of the new consensus index with respect to the previous consensus index for alternative x_i under criteria c_j for expert e_s will be:

$$\overline{CTE}_{ij}^h - CTE_{ij}^h = \delta \cdot (1 - w_T^s) \cdot \left| r_{ij}^s - \overline{r}_{ij} \right|.$$

Equality between the new consensus index and the previous one happens only when parameter $\delta = 0$, i.e when no changes are made to the original assessment for alternative x_i under criteria c_j. Obviously, the consensus index does not change if original assessment is not changed. In conclusion, experts e_s new consensus index at element level is monotonic increasing with respect to parameter δ when recommended advices values are implemented. □

Proposition 2 can be used to implement a policy of implementation of recommended advices with one expert at a time starting with the most inconsistent one. The most common policy as explained before is to recommend advices on changes of assessment at elements level to all those experts with a consistency index below the threshold value. When this is the case, for those experts that implement recommended advice values, e_s, we can obtain the range of the difference between their new consensus index with respect to their previous consensus index for alternative x_i under criteria c_j. This is presented in the following:

***Proposition* 3.** *The difference in absolute value between the new consensus index with respect to their previous consensus index for alternative x_i under criteria c_j after inconsistent experts implement recommended changes are:*

- *For inconsistent experts:* $\left|\overline{CTE}_{ij}^{s} - CTE_{ij}^{s}\right| \leq \delta \cdot \left[\left|\sum_{h=1}^{l} w_T^h \cdot \left(r_{ij}^h - \overline{r}_{ij}\right)\right| + 1\right]$
- *For consistent experts:* $\left|\overline{CTE}_{ij}^{q} - CTE_{ij}^{q}\right| \leq \delta \cdot \left|\sum_{h=1}^{l} w_T^h \cdot \left(r_{ij}^h - \overline{r}_{ij}\right)\right|$

Proof. After the experts implement recommendation advice, the assessment values for alternative x_i under criteria c_j can be divided into two groups: the set of new assessment values $\{(rr_{ij}^s) | rr_{ij}^s = (1-\delta)\cdot r_{ij}^s + \delta \cdot \overline{r}_{ij}; s = 1, \cdots, l\}$ implemented by inconsistent experts, and the set of original assessment values $\{(rr_{ij}^h) | r_{ij}^q = r_{ij}^q; q = l+1, \cdots, k\}$ corresponding to the consistent experts. Using these two sets of assessment values, we have:

$$
\begin{aligned}
\overline{rr}_{ij} &= \sum_{h=1}^{l} w_T^h \cdot rr_{ij}^h + \sum_{h=l+1}^{k} w_T^h \cdot r_{ij}^h \\
&= \sum_{h=1}^{l} w_T^h \cdot [(1-\delta) \cdot r_{ij}^h + \delta \cdot \overline{r}_{ij}] + \sum_{h=l+1}^{k} w_T^h \cdot r_{ij}^h \\
&= \overline{r}_{ij} - \sum_{h=1}^{l} w_T^h \cdot \delta \cdot \left(r_{ij}^h - \overline{r}_{ij}\right)
\end{aligned}
$$

- For $s \in \{1, \cdots, l\}$,it is:

$$
\begin{aligned}
\overline{CTE}_{ij}^{s} &= 1 - d(rr_{ij}^s - \overline{rr}_{ij}) \\
&= 1 - \left|[(1-\delta) \cdot r_{ij}^s + \delta \cdot \overline{r}_{ij}] - \left[\overline{r}_{ij} - \sum_{h=1}^{l} w_T^h \cdot \delta \cdot \left(r_{ij}^h - \overline{r}_{ij}\right)\right]\right| \\
&= 1 - \left|(1-\delta) \cdot \left(r_{ij}^s - \overline{r}_{ij}\right) + \sum_{h=1}^{l} w_T^h \cdot \delta \cdot \left(r_{ij}^h - \overline{r}_{ij}\right)\right|
\end{aligned}
$$

– Applying that $|a+b| \geq |a| - |b|$ we have

$$\begin{aligned}
\overline{CTE}_{ij}^s &\leq 1-(1-\delta)\cdot\left|r_{ij}^s-\overline{r}_{ij}\right|+\delta\cdot\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right| \\
&= \delta+(1-\delta)-(1-\delta)\cdot\left|r_{ij}^s-\overline{r}_{ij}\right|+\delta\cdot\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right| \\
&= \delta+(1-\delta)\cdot\left[1-\left|r_{ij}^s-\overline{r}_{ij}\right|\right]+\delta\cdot\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right| \\
&\leq 1-d(r_{ij}^s,\overline{r}_{ij})+\delta\cdot\left[\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right|+1\right]
\end{aligned}$$

Thus,

$$\overline{CTE}_{ij}^s - CTE_{ij}^s \leq \delta\cdot\left[\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right|+1\right]$$

– Applying that $|a+b| \leq |a| + |b|$ we have

$$\begin{aligned}
\overline{CTE}_{ij}^s &\geq 1-(1-\delta)\cdot\left|r_{ij}^s-\overline{r}_{ij}\right|-\delta\cdot\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right| \\
&\geq 1-\left|r_{ij}^s-\overline{r}_{ij}\right|-\delta\cdot\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right| \\
&\geq 1-d(r_{ij}^s,\overline{r}_{ij})-\delta\cdot\left[\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right|+1\right]
\end{aligned}$$

Thus,

$$-\delta\cdot\left[\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right|+1\right] \leq \overline{CTE}_{ij}^s - CTE_{ij}^s$$

We conclude that when $s \in \{1,\cdots,l\}$ it is:

$$\left|\overline{CTE}_{ij}^s - CTE_{ij}^s\right| \leq \delta\cdot\left[\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right|+1\right]$$

- For $q \in \{l+1,\cdots,k\}$ it is:

$$\begin{aligned}
\overline{CTE}_{ij}^q &= 1-d(rr_{ij}^q-\overline{rr}_{ij}) \\
&= 1-\left|r_{ij}^q-\left[\overline{r}_{ij}-\sum_{h=1}^{l} w_T^h\cdot\delta\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right]\right| \\
&= 1-\left|\left(r_{ij}^q-\overline{r}_{ij}\right)+\sum_{h=1}^{l} w_T^h\cdot\delta\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right|
\end{aligned}$$

Similarly to the previous case, we can conclude that when $q \in \{l+1,\cdots,k\}$ it is:

$$\left|\overline{CTE}_{ij}^q - CTE_{ij}^q\right| \leq \delta\cdot\left|\sum_{h=1}^{l} w_T^h\cdot\left(r_{ij}^h-\overline{r}_{ij}\right)\right|$$

□

Thus the range of the difference in absolute value between the consensus index at the alternatives level in subsequent steps following the implementation of the advice rules decreases when parameter δ decreases, being zero when $\delta = 0$. However, we cannot guarantee that $\overline{CTE}_{ij}^{h}$ is bigger than CTE_{ij}^{h}. Also, we cannot guarantee that $\overline{CTE}_{ij}^{h}$ increases with respect to δ.

Proposition 3 states clearly that the range of the difference in absolute value between the consensus degree at elements level when only inconsistent experts change their values is monotonic increasing with respect to the parameter δ, and therefore there is much room of improvement of consensus the bigger δ is chosen. This would be used as evidence that high values of δ push inconsistent experts towards the group and therefore will increase the group consensus at the cost of the individual independence, which is kept the lower the parameter δ is selected. Thus, it is up to the experts to willingly propose a δ parameter that represents his/her own compromise and an acceptable balance for him/her between the group consensus and his/her own independence [6, 8, 10, 11]. To support inconsistent experts in selecting the minimum δ value that place them as consistent experts in the following round, a visual adoption mechanism for recommendation advices with three steps is given below and illustrated in Fig 6.

Step 1. Choose different values for parameter δ from zero to one (δ_i), compute corresponding consensus indexes and visualise simulation of new consensus state (Fig 6(a)).

Step 2. Applying a what-if scenario analysis, the experts can easily compute and see their consensus status with respect to the rest of the group by choosing different parameters δ_i as per Table 1, and then select the minimum value δ_{min} that satisfies the consensus threshold criterion (Fig (6(b)).

Step 3. Using δ_{min} and Expression (19), visual adoption mechanism for recommendation produces personalised advice for experts according to their own willingness.

4.4. Selection Process

Each alternative is associated a final weighted criteria collective assessments

$$\overline{r}_i = \sum_{j=1}^{n} w_j \cdot \overline{r}_{ij} \tag{20}$$

Their corresponding trust degrees can be used as final score values to produce a total ordering of alternatives from which to select the best alternative.

5. Framework of the visual interaction model for consensus in SN-GDM with trust propagation

5.1. Description of the visual interaction model for consensus in SN-GDM with trust propagation problem and its components

The decision problem is how to achieve consensus in selecting the best alternative(s) from a set of feasible ones according to the evaluations provided by a group of experts socially networked with (possibly incomplete) trust statements (depicted in Figure 4). The following notations are used to denote the sets and variables used in the proposed visual interaction consensus model with trust propagation:

- $X = \{x_1, x_2, \ldots, x_m\}$ is the set of n alternatives.
- $E = \{e_1, e_2, \ldots, e_k\}$ is the set of k experts socially networked with (incomplete) trust values; w_h is the importance degree of h-th expert, which is assumed to verify $\sum_{h=1}^{k} w_h = 1,\ w_h \in [0,1]$, but it is unknown.
- $C = \{c_1, c_2, \ldots, c_n\}$ is the set of m criteria for evaluating alternatives, where ω_j is the associated weight of criterion c_j, which verify $\sum_{j=1}^{n} \omega_j = 1,\ \omega_j \in [0,1], j = 1, 2, \ldots, n$.
- $\{R^{(h)} = (r_{ij}^{(h)})_{m \times n}; h = 1, 2, \ldots, k\}$ is a collection of decision matrices given by the set of experts E, where $r_{ij}^{(h)}$ is the expert e_h evaluation of how well alternative x_i fulfil criterion c_j.

5.2. Decision process of the visual interaction model for consensus in SN-GDM with trust propagation

Step 1. Experts in the networked group express their original (possibly incomplete) trust opinions on others experts in the group, and (possibly incomplete/not fully connected) trust network is constructed (Figure 1). If required, trust propagation operator P_D (10) is applied to establish the complete (fully connected) trust network (Figure 3).

Step 2. Trust scores of each expert are computed applying (13) and used to assign importance degrees to each expert applying (14). Individual decision matrices are aggregated into a collective one $\overline{R} = (\overline{r}_{ij})_{m \times n}$ applying (15).

Step 3. Three different levels of consensus indexes for each expert are computed using (16), (18) and (17), respectively. Set a predefined threshold γ to create a visual representation of all experts consensus level with respect to the threshold value and to identify experts contributing less to group consensus.

Step 4. Activate recommendation mechanism to produce advice (19) to experts identify in previous step. The visual adoption mechanism for determining the appropriate parameter δ (Fig 6) is provided.

Step 5. Experts implement their new preference values following the provided recommendations and all the individual decision matrices are aggregated into a collective one $\overline{\overline{R}}$. If consensus degree threshold γ is achieved, then proceed to next step, otherwise go to Step 3.

Step 6. Selection process is activated by aggregating the collective assessments of each alternative with a final weighted criteria using selection (20) and producing final ranking of alternatives using (4).

6. Numerical example

A company needs to select the most appropriate cloud service supplier from four possible alternatives: $\{x_1, x_2, x_3, x_4\}$. Four criteria $\{c_1, c_2, c_3, c_4, c_5\}$ are considered: Performance; Security; Privacy; Usability and Accessibility, with associated weighting vector $\omega = (0.25, 0.40, 0.15, 0.20)^T$. A group of experts $\{e_1, e_2, e_3, e_4, e_5\}$ from five different departments with following priori trust relationship as depicted in Figure 1.

Step 1. With the original incomplete trust network (Figure 1), the complete trust network (Figure 3) is constructed using the trust propagation operator P_D (10).

Step 2. According to expression (12), the following TFs of experts are obtained:

$$\lambda_1 = [0.69, 0.56]; \quad \lambda_2 = [0.61, 0.46]; \quad \lambda_3 = [0.41, 0.29]; \quad \lambda_4 = [0.44, 0.39]; \quad \lambda_5 = [0.54, 0.38].$$

The TSs of expert are:

$$TS_1 = 0.565; \quad TS_2 = 0.575; \quad TS_3 = 0.560; \quad TS_4 = 0.525; \quad TS_5 = 0.580.$$

It is:

$$\lambda_{\sigma(1)} = \lambda_5; \quad \lambda_{\sigma(2)} = \lambda_2; \quad \lambda_{\sigma(3)} = \lambda_1; \quad \lambda_{\sigma(4)} = \lambda_3; \quad \lambda_{\sigma(5)} = \lambda_4$$

Using the BUM function $Q(r) = r^{2/3}$, the following importance weights are derived:

$$w_T^{\sigma(1)} = 0.35; \quad w_T^{\sigma(2)} = 0.20; \quad w_T^{\sigma(3)} = 0.17; \quad w_T^{\sigma(4)} = 0.15; \quad w_T^{\sigma(5)} = 0.13$$

The corresponding experts weights are:

$$w_T^1 = 0.17; \quad w_T^2 = 0.20; \quad w_T^3 = 0.15; \quad w_T^4 = 0.13; \quad w_T^5 = 0.35$$

Assume the five experts give the evaluation of the four cloud service suppliers via the following decision matrices:

$$R^1 = \begin{pmatrix} (0.5, 0.5) & (0.3, 0.5) & (0.4, 0.6) & (0.5, 0.6) \\ (0.5, 0.6) & (0.6, 0.4) & (0.5, 0.6) & (0.8, 0.7) \\ (0.4, 0.7) & (0.2, 0.5) & (0.5, 0.5) & (0.4, 0.5) \\ (0.3, 0.6) & (0.4, 0.4) & (0.4, 0.6) & (0.5, 0.5) \end{pmatrix}$$

$$R^2 = \begin{pmatrix} (0.5,0.5) & (0.4,0.5) & (0.3,0.6) & (0.4,0.6) \\ (0.5,0.7) & (0.5,0.5) & (0.5,0.8) & (0.6,0.6) \\ (0.6,0.4) & (0.4,0.5) & (0.5,0.5) & (0.4,0.6) \\ (0.4,0.5) & (0.3,0.5) & (0.5,0.6) & (0.5,0.5) \end{pmatrix}$$

$$R^3 = \begin{pmatrix} (0.5,0.5) & (0.3,0.6) & (0.4,0.6) & (0.4,0.6) \\ (0.4,0.6) & (0.5,0.5) & (0.3,0.5) & (0.4,0.7) \\ (0.5,0.5) & (0.5,0.6) & (0.5,0.5) & (0.6,0.5) \\ (0.7,0.5) & (0.6,0.5) & (0.3,0.6) & (0.5,0.5) \end{pmatrix}$$

$$R^4 = \begin{pmatrix} (0.6,0.5) & (0.7,0.6) & (0.5,0.5) & (0.6,0.1) \\ (0.4,0.6) & (0.5,0.5) & (0.6,0.6) & (0.5,0.4) \\ (0.2,0.6) & (0.5,0.7) & (0.7,0.3) & (0.8,0.7) \\ (0.2,0.8) & (0.3,0.7) & (0.5,0.4) & (0.4,0.6) \end{pmatrix}$$

$$R^5 = \begin{pmatrix} (0.4,0.2) & (0.4,0.6) & (0.6,0.8) & (0.5,0.2) \\ (0.5,0.8) & (0.4,0.2) & (0.5,0.3) & (0.4,0.7) \\ (0.3,0.4) & (0.4,0.6) & (0.6,0.8) & (0.7,0.3) \\ (0.4,0.6) & (0.7,0.4) & (0.5,0.6) & (0.6,0.7) \end{pmatrix}$$

According to expression (15), the trust based collective decision matrix is calculated:

$$\overline{R} = \begin{pmatrix} (0.478,0.395) & (0.422,0.563) & (0.448,0.657) & (0.478,0.395) \\ (0.487,0.690) & (0.482,0.378) & (0.513,0.520) & (0.551,0.641) \\ (0.409,0.492) & (0.379,0.576) & (0.561,0.579) & (0.557,0.476) \\ (0.357,0.591) & (0.457,0.474) & (0.483,0.574) & (0.522,0.583) \end{pmatrix}$$

Step 3. The consensus indexes at the pair of alternative-criterion level are:

$$CTE^1 = \begin{pmatrix} 0.937 & 0.908 & 0.948 & 0.887 \\ 0.949 & 0.930 & 0.954 & 0.846 \\ 0.892 & 0.873 & 0.930 & 0.910 \\ 0.967 & 0.935 & 0.946 & 0.948 \end{pmatrix} \quad CTE^2 = \begin{pmatrix} 0.937 & 0.958 & 0.898 & 0.859 \\ 0.989 & 0.930 & 0.854 & 0.956 \\ 0.859 & 0.952 & 0.930 & 0.860 \\ 0.933 & 0.909 & 0.979 & 0.948 \end{pmatrix}$$

$$CTE^3 = \begin{pmatrix} 0.937 & 0.921 & 0.948 & 0.859 \\ 0.912 & 0.930 & 0.884 & 0.890 \\ 0.951 & 0.928 & 0.9300 & 0.967 \\ 0.783 & 0.916 & 0.895 & 0.948 \end{pmatrix} \quad CTE^4 = \begin{pmatrix} 0.887 & 0.843 & 0.896 & 0.792 \\ 0.912 & 0.930 & 0.917 & 0.854 \\ 0.842 & 0.878 & 0.791 & 0.767 \\ 0.817 & 0.809 & 0.905 & 0.931 \end{pmatrix}$$

$$CTE^5 = \begin{pmatrix} 0.864 & 0.971 & 0.853 & 0.892 \\ 0.938 & 0.870 & 0.884 & 0.895 \\ 0.899 & 0.978 & 0.870 & 0.841 \\ 0.974 & 0.842 & 0.979 & 0.921 \end{pmatrix}$$

The consensus indexes at the alternatives level are:

$$CTA^1 = (0.919, 0.919, 0.901, 0.947); \quad CTA^2 = (0.913, 0.932, 0.900, 0.942);$$
$$CTA^3 = (0.916, 0.905, 0.944, 0.885); \quad CTA^4 = (0.854, 0.903, 0.819, 0.865);$$
$$CTA^5 = (0.919, 0.919, 0.901, 0.949).$$

The individual consensus indexes at the decision matrix level are:

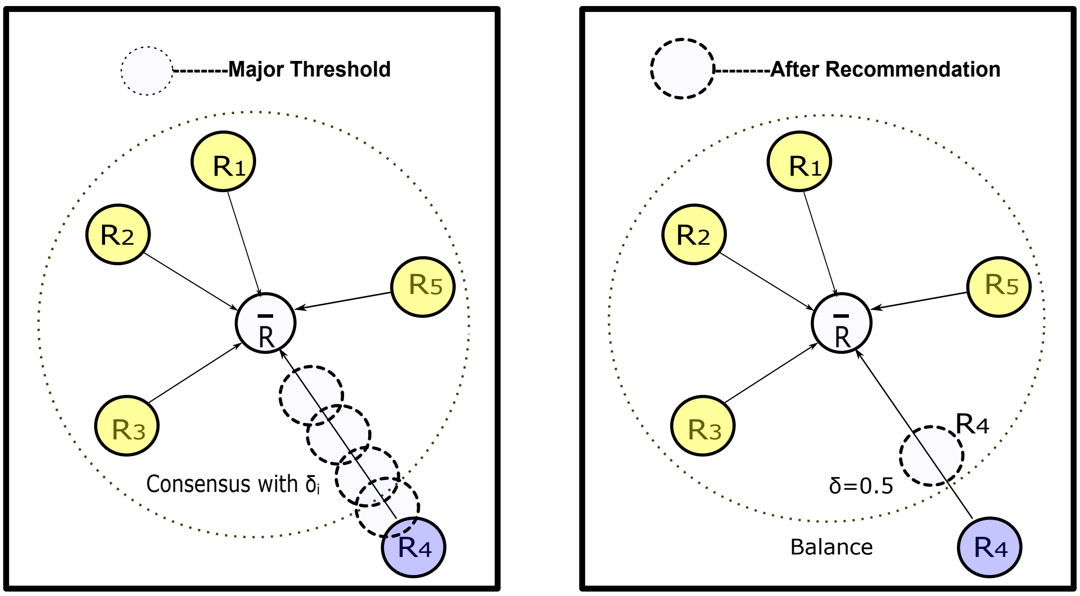
$$CI^1 = 0.922,\ CI^2 = 0.921,\ CI^3 = 0.912,\ CI^4 = 0.860,\ CI^5 = 0.903.$$

With a set threshold value of $\gamma = 0.9$, the recommendation simulation is activated to assist expert e_4 to modify his/her some of original preference values to increase his/her consensus status. The following APS set is obtained:

$$APS = \{(4,1,1), (4,1,2), (4,1,3), (4,1,4), (4,3,1), (4,3,2), (4,3,3), (4,3,4), (4,4,1), (4,4,2)\}.$$

The generation of advice is activated.

Step 4. Choosing different δ_i values, we visualise the corresponding consensus indexes at the decision matrix level of e_4 in Fig 6, and produce them in Table 1.



(a) Visual recommendation simulation with δ_i (b) Balance δ_i between individual and group

Figure 6: Visual representation of consensus levels after implementation of recommended values

Table 1: Consensus index at decision matrix level for different δ_i values

δ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
CI	0.860	0.870	0.878	0.888	0.897	0.906	0.915	0.924	0.934	0.943	0.952

Obviously, for $\delta \geq 0.5$ the new consensus index is above the consensus threshold value 0.9 is reached. Therefore, the minimum value that will place expert e_4 above the consensus threshold with lower change variation from his/her original assessment will be $\delta = 0.5$. If such value of parameter δ is chosen by expert e_4, the system will generate the following recommended values to implement:

To increase your consensus with the rest of the group, your trust assessment for alternative

- x_1 under criteria c_1 should to be closer to $(0.539, 0.447)$.
- x_1 under criteria c_2 should to be closer to $(0.561, 0.582)$.
- x_1 under criteria c_3 should to be closer to $(0.474, 0.579)$.
- x_1 under criteria c_4 should to be closer to $(0.539, 0.247)$.
- x_3 under criteria c_1 should to be closer to $(0.305, 0.546)$.
- x_3 under criteria c_2 should to be closer to $(0.440, 0.638)$.
- x_3 under criteria c_3 should to be closer to $(0.631, 0.440)$.
- x_3 under criteria c_4 should to be closer to $(0.679, 0.588)$.
- x_4 under criteria c_1 should to be closer to $(0.279, 0.696)$.
- x_4 under criteria c_2 should to be closer to $(0.379, 0.587)$.

Step 5. Once expert e_4 changes his/her trust decision matrix with the above recommended values, a new consensus process round takes place. The resultant new collective trust decision making matrix is

$$\overline{\overline{R}} = \begin{pmatrix} (0.470, 0.388) & (0.404, 0.561) & (0.445, 0.667) & (0.470, 0.414) \\ (0.487, 0.690) & (0.482, 0.378) & (0.513, 0.520) & (0.551, 0.641) \\ (0.423, 0.485) & (0.371, 0.568) & (0.552, 0.597) & (0.541, 0.461) \\ (0.367, 0.577) & (0.467, 0.459) & (0.483, 0.574) & (0.522, 0.583) \end{pmatrix}$$

The new consensus indexes would become: $CI^1 = 0.921$, $CI^2 = 0.923$, $CI^3 = 0.912$, $CI^4 = 0.906$, $CI^4 = 0.903$, which are all above the threshold value $\gamma = 0.9$. This means the selection process is activated to derive the group solution of consensus.

Step 6. The criteria weighting vector $\omega = (0.25, 0.40, 0.15, 0.20)^T$ produces the following collective overall evaluation values of alternatives:

$$\overline{r}_1 = (0.440, 0.504),\ \overline{r}_2 = (0.502, 0.530),\ \overline{r}_3 = (0.445, 0.530),\ \overline{r}_4 = (0.456, 0.531)$$

Their corresponding trust degrees $TD_1 = 0.472$, $TD_2 = 0.516$, $TD_3 = 0.487$, $TD_4 = 0.493$ lead to the final ordering $x_2 \succ x_4 \succ x_3 \succ x_1$, which makes alternative x_2 to be selected as the group solution of consensus.

7. Conclusion

In this article, a novel visual interactive method for consensus in SN-GDM has been presented. This method consists of three main modules: a dual trust propagation, a trust based recommendation and a visual adoption mechanism. The following main advantages with respect to other consensus models proposed in the literature are highlighted:

1. The novel dual trust propagation operator is based on t-norms and t-conorms (Einstein product and Einstein sum used as example in this paper), respectively. It propagates trust in a reasonable way: trust values decrease while distrust values increase. Therefore, in comparison with previous trust propagation operators [31, 38, 41], information attenuation caused by the trusted third parters (TTP) is appropriately addressed and modelled.

2. The trust based recommendation produces advices for inconsistent experts to increase their respective consensus level. The propagated trust by TTP is used to build the trust relationship within the group of experts and, in turn, to obtain their corresponding trust score (TS). TS reflects the actual reputation of experts that derived from historic interaction information in the Social Network, and it can be regarded as a reliable source to assign their corresponding importance degrees. The recommendation advice is thus generated in a reasonable way: the higher the trust score of an expert, the more important the opinion associated to that expert is. Trust relationship between experts in a group is fully exploited to produce recommendations to experts while existing consensus models neglect this usage [14, 44]. Hence, the trust based recommendation is more appropriate to deal with SN-GDM.

3. The reaching consensus visual adoption mechanism supports inconsistent experts by providing them with visual representations of current and future consensus positions before and after recommendation advices are implemented. In the light of this visual extra information, they can see how to reach the consensus threshold value with minimum changes cost, and consequently a balance between group consensus and individual independence is possible to be achieved. Therefore, the advantage of the proposed visual adoption mechanism is that inconsistent experts might be more willing to implement the recommendation advices to increase consensus without being forced to, an aspect existing consensus models do not appropriately address [14, 44].

Although this visual interaction model includes a consensus simulation with different feedback parameters to allow experts to reach consensus, it is not able, however, to determine the exact value of the parameter δ that would provide the optimal balance between group consensus and individual independence, i.e. the minimum change of original opinion required to contribute to consensus at the threshold value established a priori by the whole group of experts. This is an aspect that deserves further work and that we intent to address in future.

Acknowledgements

The authors would like to acknowledge the anonymous referees for providing constructive comments and suggestions. Also acknowledge is given to the financial support received from National Natural Science Foundation of China (NSFC) (No.71571166); Zhejiang Provincial Natural Science Foundation of China (No.LY15G010003); Zhejiang Provincial Philosophy and Social Science Foundation-Zhijiang Young Talent of Social Science (No.16ZJQN046YB); and FEDER funds from projects TIN2013-40658-P and TIN2016-75850-R.

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